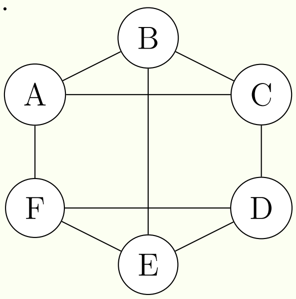
# Grade 7/8 Math Circles
## February 14/15/16, 2023
## Graph Theory - Solutions

## Exercise Solutions

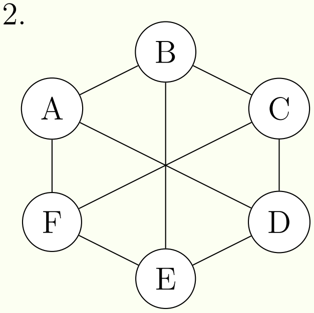
**Exercise 1**

Match the sets of vertices and edges to the correct visual picture of the graph.

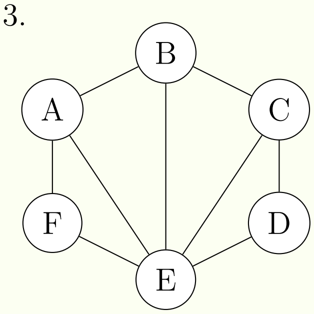
1.




2.




3.




(a) $V = \{A, B, C, D, E, F\}$
$E = \{AB, BC, CD, DE, EF, AF, AD, CF\}$

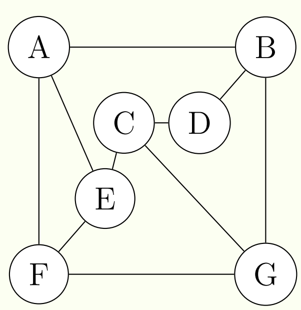
(b) $V = \{A, B, C, D, E, F\}$
$E = \{AB, AE, BC, CD, DE, EC, EF, AF\}$

(c) $V = \{A, B, C, D, E, F\}$
$E = \{AB, BC, CD, DE, CA, DF, EF, AF\}$

**Solution**

1. (c) 2. (a) 3. (b)

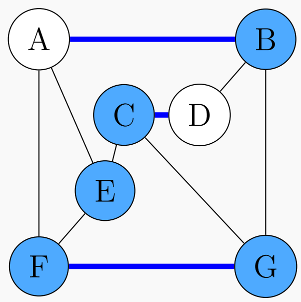
**Exercise 2**



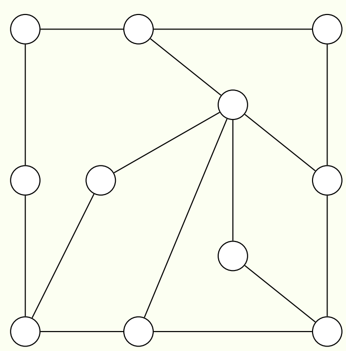
1. Find a different valid matching for the graph above. Show this in a drawing and in proper notation.
2. Find a different valid cover for the graph above. Show this in a drawing and in proper notation.

**Solution**

Solutions may vary, but a possible solution is $\mathcal{M} = \{AB, CD, FG\}$ and the vertices in the cover are $\{B, C, E, F, G\}$.




**Exercise 3**

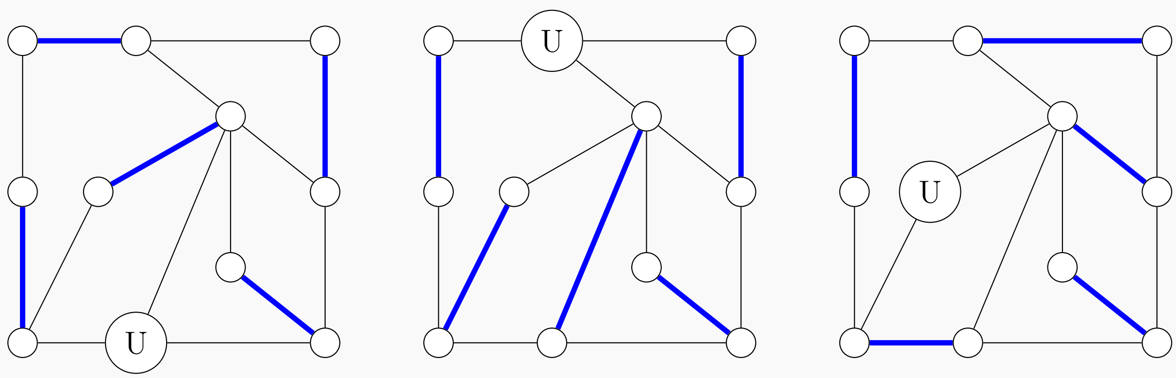


1. A maximum matching is a matching with the largest possible size for a given graph. What is the size of the maximum matching of graph above?
2. A minimum cover is a cover with the smallest possible size for a given graph. What is the size of the minimum cover of the graph above?

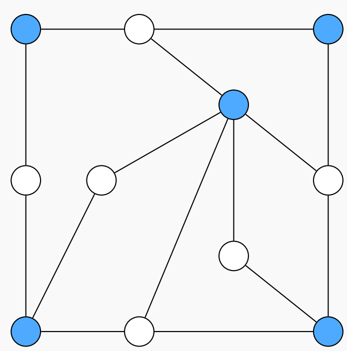
We will look at these two ideas later when we talk about bipartite graphs.

**Solution**

1. The size of the maximum matching is 5. We will explain why this is the case in Example E. Below are a few possible maximum matchings (in blue). Notice that no more edges can be added to the matching, because to do so would cause two edges to share an endpoint. The only unmatched vertex, $U$, is highlighted.

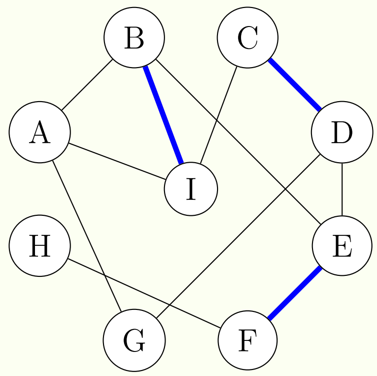



2. The size of the minimum cover is 5. We will also explain why this is the case with a theorem. There is actually only one valid minimum cover for this graph. This is not always the case, but notice that since there are 5 edges connected to the blue vertex in the middle, it makes sense that that vertex would probably need to be in the cover to minimize the number of vertices.



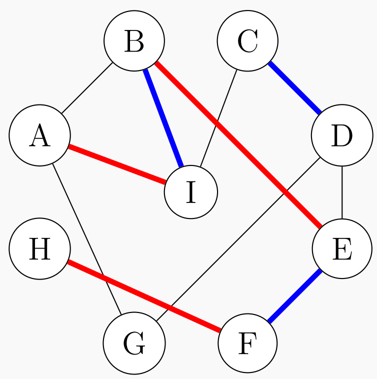
**Exercise 4**

Given the following graph and matching (in blue), show that the matching is not maximal by finding an augmenting path.




**Solution**

The beginning of an augmenting path must start with an unmatched vertex. Since $EF$ is in the matching, we can start our augmenting path at $H$. So we have the following augmenting path highlighted in red and blue:



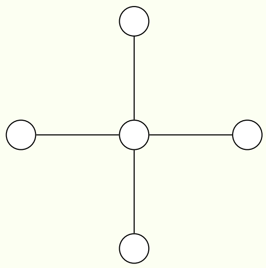

Since we have an augmenting path, we can apply Berge's Theorem to conclude that the given matching is NOT maximal.

Notice that the sequence $HF$,$FE$,$ED$,$DC$,$CI$,$IB$,$BA$ is also a valid augmenting path.

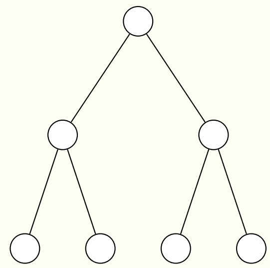
**Exercise 5**

Classify the following as a bipartite or non-bipartite graph.

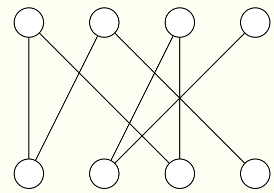
1.



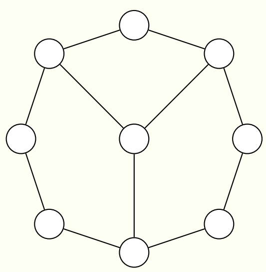
2.



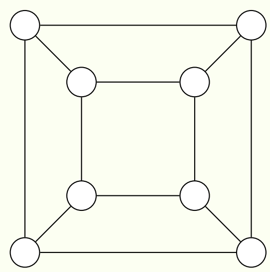
3.



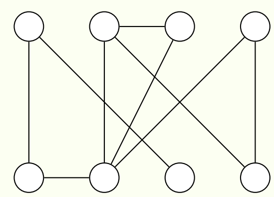
4.



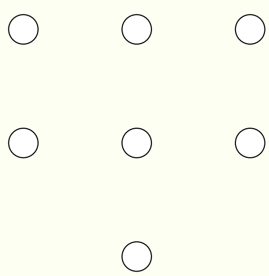
5.



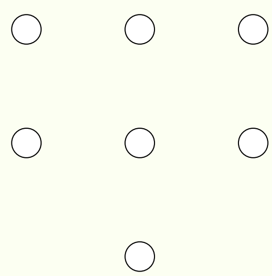
6.



Draw edges between the vertices to make a bipartite and non-bipartite graph.



bipartite



non-bipartite

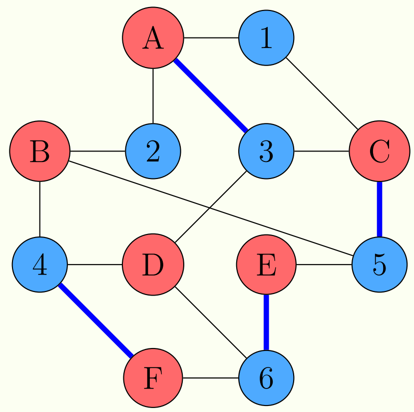
**Solution**

1. bipartite 2. bipartite 3. bipartite 4. non-bipartite 5. bipartite 6. non-bipartite

Answers may vary for the drawing.

**Exercise 6**

Suppose that the blue edges are in the matching. Find a maximum matching of the graph below by using the Hopcroft-Karp Algorithm. The two partitions are $X = \{A, B, C, D, E, F\}$ and $Y = \{1, 2, 3, 4, 5, 6\}$.

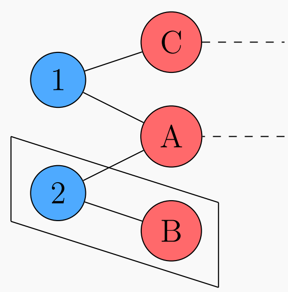



**Solution**

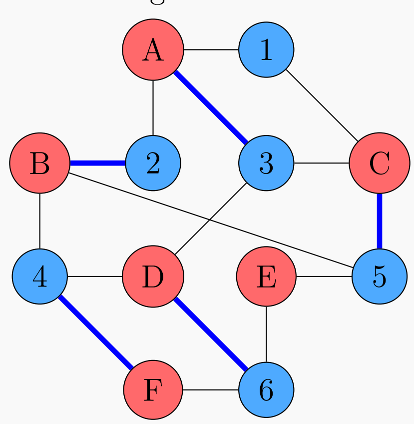
We first start by choosing a partition. Let us choose $Y$ (the blue numbers) as our starting partition.

Iteration 1:

The unmatched vertices in $Y$ are 1 and 2. So we draw the following alternating tree.

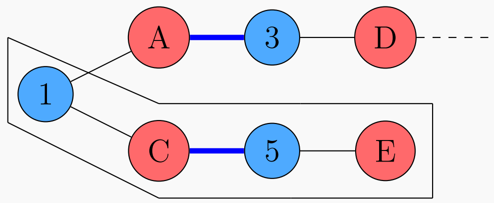



Since $A$ and $C$ are both matched vertices and $B$ is unmatched, we only have one possible augmenting path, $B2$. Since there were no matched edges in our augmenting path, swapping the edges in this path only results in adding $B2$ to our matching.

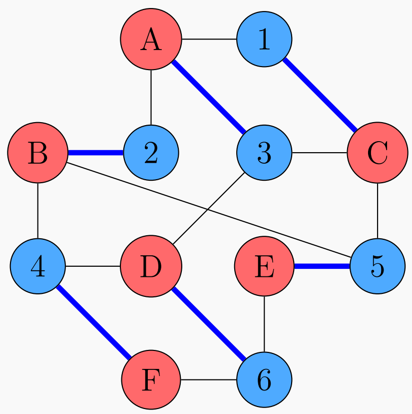


Iteration 2:

The only unmatched vertex in $Y$ is 1. So we draw the following alternating tree.



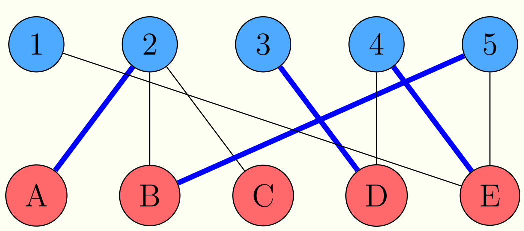

This gives us the highlighted augmenting path. Swapping the edges in this path then us the following updated graph:




Since we don't have any more unmatched vertices in $Y$, we stop here. We find a maximum matching $\mathcal{M} = \{A3, B2, C1, D6, E5, F4\}$.

**Exercise 7**

Given the following bipartite graph, with partitions $X = \{A, B, C, D, E\}$ and $Y = \{1, 2, 3, 4, 5\}$, and the following maximum matching (in blue), use König's Theorem and the Minimum Cover Rule to find a minimum cover.



**Solution**

Since this graph is bipartite and we are given a maximum matching, König's Theorem tells us that there is a minimum cover that is the same size as our maximum matching.

We start by building $U$ and $Z$. In $X$, only vertex $C$ is unmatched. So $U = \{C\}$. Then we consider the alternating paths from $C$, which is only the sequence $\{C, 2, A\}$. So $Z = \{A, C, 2\}$. This means that:

$$\text{vertices in } X \text{ but not in } Z = \{B, D, E\}$$

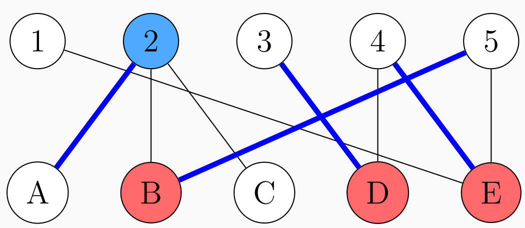
and

$$\text{vertices in both } Y \text{ and } Z = \{2\}$$

So our minimum cover by the Minimum Cover Rule is:

$$\begin{aligned}\text{minimum cover} &= (\text{vertices in } X \text{ but not in } Z) \text{ and } (\text{vertices in both } Y \text{ and } Z)\\ &= \{B, D, E\} \text{ and } \{2\}\\ &= \{B, D, E, 2\}\end{aligned}$$

The highlighted vertices shows the minimum cover $\mathcal{C} = \{B, D, E, 2\}$




Note again that the size of our minimum cover matches the size of our maximum matching.

## Problem Set Solutions

1. Your friend says that they can draw a graph with the same number of vertices as edges. Is what your friend saying possible? Why or why not?

   *Solution*: This is possible. Consider a graph with three vertices and three edges. The only way to make a valid graph is to make a cycle of length three (triangle).

2. Another friend says that they can draw a bipartite graph with no edges. Is what your friend saying possible? Why or why not?

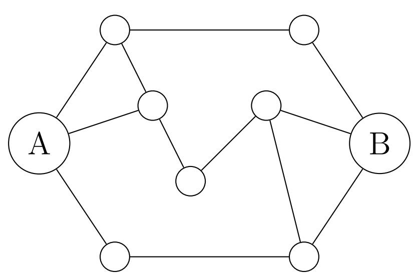
   *Solution*: In the lesson, we defined a bipartite graph as a graph that has no edges from one a vertex in a partition to another vertex in the same partition. If we choose one partition to one vertex and the other vertices to be the other partition. Since we have no edges in the graph, we guarantee that a vertex in one partition cannot have an edge to another vertex in the same partition.

3. Suppose you have the following graph.

   (a) How many possible paths are there from vertex $A$ to vertex $B$?

   (b) How many cycles are in this graph?

   (c) How many possible walks are there from vertex $A$ to vertex $B$?



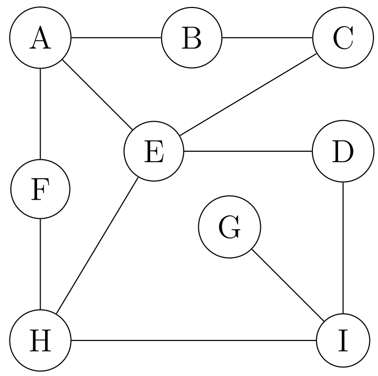

*Solution*:

(a) 9

(b) 4

(c) An infinite amount. This is because repeated vertices are allowed in a in a walk.

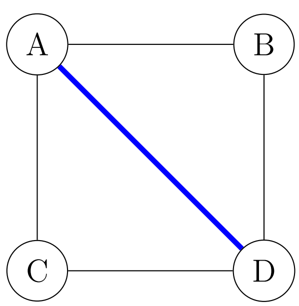
4. Show that the following graph is bipartite by separating the vertices into two partitions where a vertex in one partition does not have an edge to another vertex in the same partition. In the lesson, we learned that every bipartite graph can only contain even length cycles. Use this to check your answer.



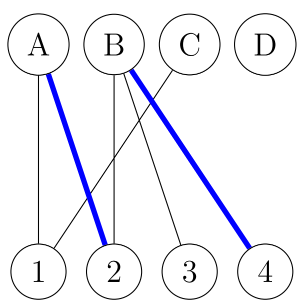

*Solution*: The two partitions can be defined as $X = \{A, C, D, G, H\}$ and $Y = \{B, E, F, I\}$. Checking our cycles, we see that there are three in the graph. They are: $\{A, E, F, H\}$, $\{A, B, C, E\}$, and $\{E, D, H, I\}$. We can see that all of these are cycles of length four, which checks out with our characterization of bipartite graphs.

5. Given the following graphs and their follow matchings, use Berge's Theorem to show whether the matchings are maximum or not.

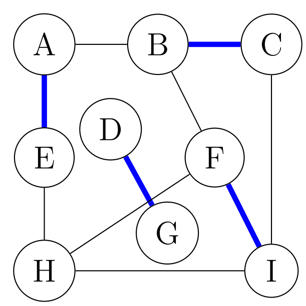
(a)




(b)




(c)



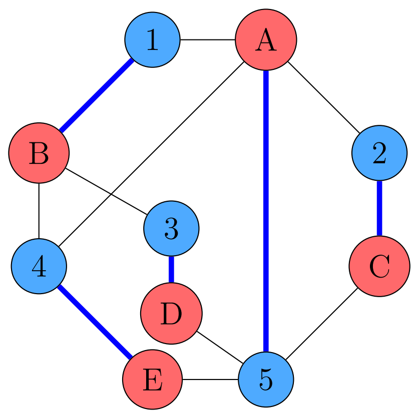

*Solution*:

(a) An augmenting path is $BA, AD, DC$, so by Berge's Theorem, the matching is not maximum.

(b) A trivial augmenting path is $C1$, so by Berge's Theorem, the matching is not maximum.

(c) Since $H$ is the only unmatched vertex, we can guarantee that we cannot find any augmenting path. So by Berge's Theorem, the matching is maximum.

6. Given the following bipartite graph and maximum matching, use the Minimum Cover Rule to find a minimum cover.




*Solution*: Let us first define our partitions $X$ and $Y$.

$X = \{A, B, C, D, E\}$ and $Y = \{1, 2, 3, 4, 5\}$

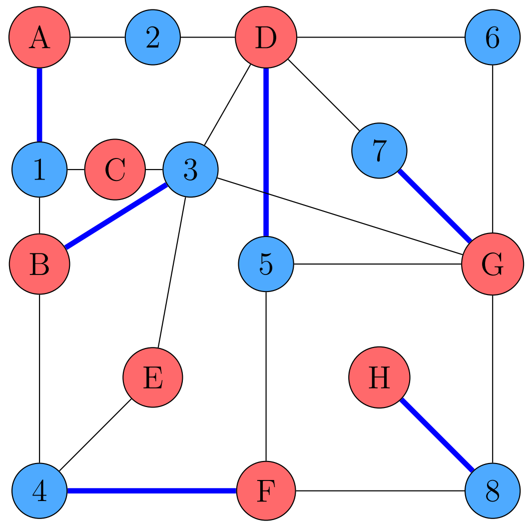
There are no unmatched vertices in $X$, so $U = \emptyset$ (is empty). Since $U$ is empty, there are no vertices that are connected to the vertices in $U$. So $Z = \emptyset$ (is empty) as well.
By the Minimum Cover Rule, the minimum cover is given by:

$$\begin{aligned}\text{minimum cover} &= (\text{vertices in } X \text{ but not in } Z) \text{ and } (\text{vertices in both } Y \text{ and } Z)\\ &= (\text{vertices in } X) \text{ and } (\text{no vertices})\\ &= \{A, B, C, D, E\}\end{aligned}$$

7. Below is a bipartite graph with partitions $X = \{A, B, C, D, E, F, G, H\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and a matching in blue.

(a) Use the Hopcroft-Karp Algorithm to find a maximum matching

(b) Apply the Minimum Cover Rule to find a minimum cover.

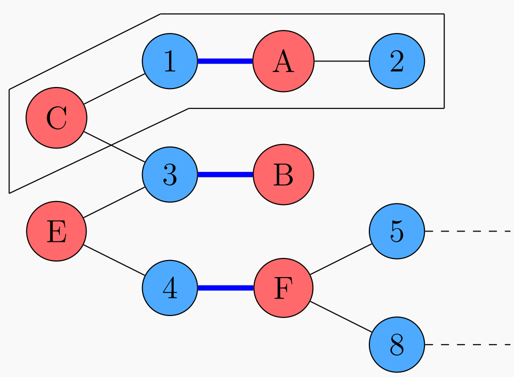



*Solution*:

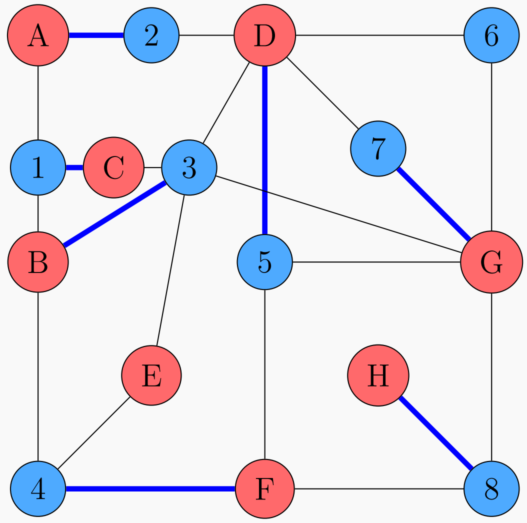
(a) First, we choose our partition to work from. Let us choose $X$ to be our starting partition.

Iteration 1:

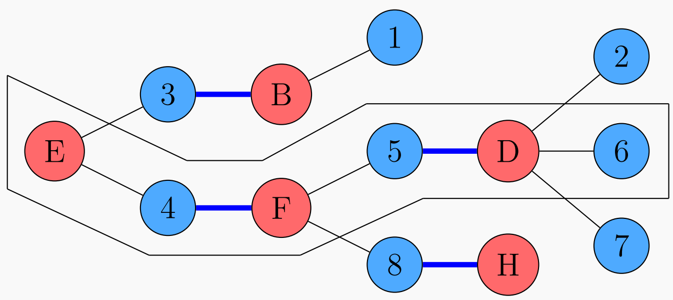
The unmatched vertices in $X$ are $C$ and $E$. So we start with those and draw the following alternating tree.



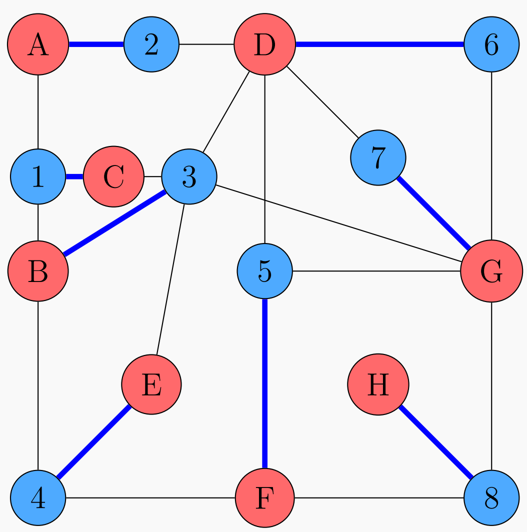

So we get $C1, 1A, A2$ as our augmenting path. Swapping the edges in the augmenting path, we get the following updated graph.




Iteration 2: Now the only unmatched vertex left in $X$ is $E$. So we start from $E$ and draw the following alternating tree.




So we get $E4, 4F, F5, 5D, D6$ as our augmenting path. Swapping the edges in the augmenting path, we get the following update graph.

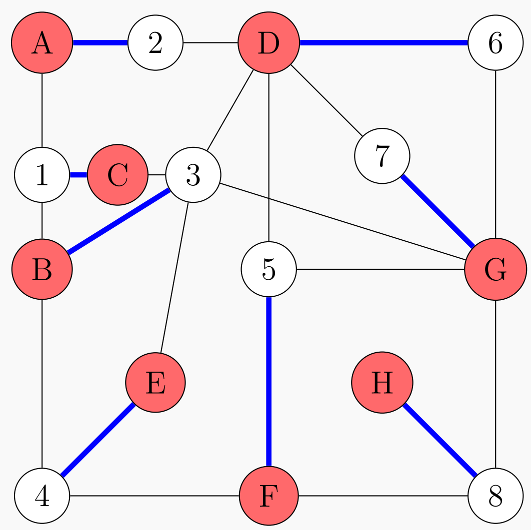


Since we have no more unmatched vertices in $X$, we stop and have our maximum matching $\mathcal{M} = \{A2, B3, C1, D6, E4, F5, G7, H8\}$.

(b) Since we have a maximum matching we can find a minimum cover of the same size by König's Theorem, and we use the Minimum Cover Rule to find such a cover.

We see that none of the vertices in $X$ are unmatched. This means that $U = \emptyset$. Since $U$ is empty, there are no alternating paths from the vertices in $U$ and so $Z = \emptyset$as well. This gives us:

$$\begin{aligned} \text{minimum cover} &= (\text{vertices in } X \text{ but not in } Z) \text{ and } (\text{vertices in both } Y \text{ and } Z) \\ &= (\text{vertices in } X) \text{ and } (\text{no vertices}) \\ &= \{A, B, C, D, E, F, G, H\} \text{ and } \emptyset \\ &= \{A, B, C, D, E, F, G, H\} \end{aligned}$$




The red vertices highlight our minimum cover $\mathcal{C} = X = \{A, B, C, D, E, F, G, H\}$. Note that if we swapped $X$ and $Y$, we would get that $\mathcal{C} = \{1, 2, 3, 4, 5, 6, 7, 8\}$.